Math 246C Lecture 16 Notes

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1 Consequences of the Uniformization Theorem

1.1 Deck transformations

We have shown the Uniformization theorem.

Theorem 1.1 (Uniformization)**.** *Let X be a simply connected Riemann surface.*

1. *If Green's function exists for X, then there is a holomorphic bijection $X \to D$.*
2. *If X is compact, then $X \cong \hat{\mathbb{C}}$.*
3. *If X is not compact and if Green's function does not exist, then $X \cong \mathbb{C}$.*

What does this say about non-simply connected Riemann surfaces?

Let X be a connected topological manifold. Let $\tilde{X}$ be the universal covering space of X with covering map $p : \tilde{X} \to X$.

Definition 1.1. We say that a homeomorphism $\varphi : \tilde{X} \to \tilde{X}$ is a **deck transformation** if $p \circ \varphi = p$.

Proposition 1.1. *The set of deck transformations is a group G which acts transitively on the fibers: if $\tilde{x}, \tilde{y} \in \tilde{X}$ such that $p(\tilde{x}) = p(\tilde{y})$, there is a unique $\varphi \in G$ such that $\varphi(\tilde{x}) = \tilde{y}$.*

Proof. The lifting criterion applied to p gives a unique $\varphi : \tilde{X} \to \tilde{X}$ such that $p \circ \varphi = p$ and $\varphi(\tilde{x}) = \tilde{y}$.

$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ \tilde{X} \arrow[r, "p"'] \arrow[ur, dashed, "\varphi"] & X \end{tikzcd}$$

φ is a homeomorphism because there is a continuous map $\psi : \tilde{X} \to \tilde{X}$ such that $P \circ \tilde{\psi} = p$ and $\psi(\tilde{y}) = \tilde{x}$. So $p \circ \varphi \circ \psi = p$ and $\varphi(\psi(\tilde{y})) = \tilde{y}$. So $\varphi \circ \psi = 1$ by the uniqueness of lifts. So φ is a deck transformation. $\square$

Proposition 1.2. *The group G acts on $\tilde{X}$ freely: for all $\varphi \in G$ with $\varphi \neq 1$, φ has no fixed points. Also, the orbits $G\tilde{x} = \{\varphi(\tilde{x}) : \varphi \in G\} = p^{-1}(p(\tilde{x}))$ are discrete, as p is a cover.*

Corollary 1.1. *The space of orbits $\tilde{X}/G$ is naturally identified with X, also topologically if $\tilde{X}/G$ is equipped with the quotient topology: $O \subseteq \tilde{X}/G$ is open iff $\pi^{-1}(P) \subseteq \tilde{X}$ is open, where $\pi : \tilde{X} \to \tilde{X}/G$ is the quotient map $\tilde{x} \mapsto G\tilde{x}$.*

1.2 Partial classification of Riemann surfaces

Let X be a Riemann surface. Then $\tilde{X}$ is a Riemann surface, and $p : \tilde{X} \to X$ is holomorphic. So every $\varphi \in G$ is holomorphic: $G \subseteq \mathrm{Aut}(\tilde{X}) = \{\text{holomorphic bijections } \tilde{X} \to \tilde{X}\}$. We have $X = \tilde{X}/G$, where by uniformization, $\tilde{X} = \hat{\mathbb{C}}$, $\mathbb{C}$, or D.

1. $\tilde{X} = \hat{\mathbb{C}}$: We have that $G \subseteq \mathrm{Aut}(\hat{\mathbb{C}}) = \{\sigma : \sigma(z) = \frac{az+b}{cz+d}, ad - bc \neq 0\}$. Every $\sigma \in \mathrm{Aut}(\mathbb{C})$ has a fixed point, so $G = \{1\}$. We get that if X is a Riemann surface with $\hat{\mathbb{C}}$ has the universal covering space, $X = \mathbb{C}$.

2. $\tilde{X} = \mathbb{C}$: We have that $G \subseteq \mathrm{Aut}(\mathbb{C}) = \{\sigma : \sigma(z) = az + b, a \neq 0, b \in \mathbb{C}\}$. The elements of G have no fixed points, so $a = 1$. We get that $G \subseteq \{\sigma : \sigma(z) = z + b, b \in \mathbb{C}\}$, the complex translations. G acts with discrete orbits, so (by a fact we will not prove here[1]) one of the following holds:

   (a) $G = \{1\}$, so $X \cong \mathbb{C}$.

   (b) $G = \{\sigma : \sigma(z) = z + n\gamma, n \in \mathbb{Z}\}$ for some $\gamma \in \mathbb{C} \setminus \{0\}$. We have a natural isomorphism $X \cong \mathbb{C}/\{z \mapsto z + n\gamma\} \cong \mathbb{C} \setminus \{0\}$ via $[z] \mapsto e^{2\pi i z/\gamma}$.

   (c) $G = \{\sigma : \sigma(z) = n\gamma + m\delta + z, n, m \in \mathbb{Z}\}$, where $\gamma, \delta \in \mathbb{C}$ are linearly independent over $\mathbb{R}$. In this case, X is isomorphic to the complex torus.

   Thus, if X is a Riemann surface with $\tilde{X} = \mathbb{C}$, then $X \cong \mathbb{C}$, $\mathbb{C} \setminus \{0\}$, or a complex torus.

3. $\tilde{X} = D$. Then $X \cong D/G$, where $G \subseteq \mathrm{Aut}(D)$ acts freely. Such subgroups are called **Fuchsian groups**. This is the general case.

1.3 Examples of applications

Example 1.1. Let M be a compact Riemann surface, and assume that there is some $f \in \mathrm{Hol}(\mathbb{C}, M)$ which is non-constant. What can be said about M? Lift f to the universal

[1] This fact has nothing to do with Riemann surfaces. We have a discrete group acting on a real vector space, so the number of generators should be $\leq$ the dimension of the vector space.

covering space:

$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ \mathbb{C} \arrow[r, "f"'] \arrow[ur, dashed, "\tilde{f}"] & X \end{tikzcd}$$

Then $\tilde{f}$ is non-constant, so $\tilde{M} \neq D$. If $\tilde{M} = \hat{\mathbb{C}}$, then either $M \cong \hat{\mathbb{C}}$ or $\tilde{M} = \mathbb{C}$ and $M \cong$ a torus.

Theorem 1.2 (Picard's little theorem)**.** *Let* $f \in \mathrm{Hol}(\mathbb{C})$ *be such that* $0, 1 \notin f(\mathbb{C})$*. Then* f *is constant.*

Proof. We can lift f:

$$\begin{tikzcd} & D \arrow[d, "p"] \\ \mathbb{C} \arrow[r, "f"'] \arrow[ur, dashed, "\tilde{f}"] & \mathbb{C} \setminus \{0, 1\} \end{tikzcd}$$

By Liouville's theorem, $\tilde{f}$ is constant. So f is constant. □

This is the end of our discussion of Riemann surfaces. If you are interested in learning more, here are books which have a modern approach to analysis on Riemann surfaces:

- S. Donaldson, Riemann surfaces.
- D. Varolin, Riemann surfaces by way of complex analytic geometry.